Julia Sets that are Full of Holes

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Fractals are often thought of as just pretty computer pictures, such as Figure 1. There are many types of fractals. We will be focusing on some Julia sets, which are fractals that belong to the field of complex dynamics. Complex dynamics uses techniques from many fields, especially complex analysis. It is a field whose first explorers included Gaston Julia and Pierre Fatou at the turn of the twentieth century, but which made its largest gains in the 1980s with the use of computers to discover the beauty of Julia sets and the Mandelbrot set.

If we look at a picture of a Julia set on the computer, it looks fairly complex. However, the computer's impression could be misleading, so we try to mathematically verify the complexity. First, we try to determine if the Julia set takes up any space, in the sense of having a Lebesgue measure greater than zero. A set which looks like a curve on the screen but takes up space would be extremely complex. However, since most known Julia sets do not take up space, we need to be able to classify the complexity by different means. We do this classification by looking at the dimension of the set.

The possibilities for the dimension of a Julia set are quite vast. There are Julia sets whose dimensions range from 0 to 2, whose pictures range from fractal "dust" to the whole Reimann sphere. With the exception of those which are the whole Reimann sphere, all Julia sets whose measure has been calculated have had measure zero; however, it has been conjectured that there are Julia sets of positive measure.

We will look at one way to show Julia sets have measure zero and that their dimension is less than two. This method shows a property called porosity and involves finding holes in our Julia sets at every scale. In other words, no matter how close we look at our Julia set, it is full of holes.

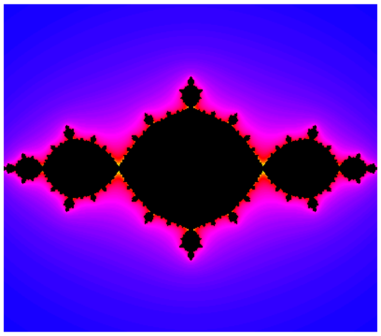

Figure 1: The Julia set for $f(z) = z^2 - 1$.

I have tried to make this as readable as possible for all mathematicians who have at least taken an undergraduate course in analysis and preferably one in complex analysis. If you have not taken a course in measure theory, feel free to think of the word measure as an idea similar to area. Good introductions for undergraduates can be found in an informal setting in Lesmoir-Gordon et al [LGRE] and in a more formal one in Devaney [De]. A good graduate level introduction to complex dynamics can be found it Carleson and Gamelin, [CG].

What is a Julia Set?

Julia sets come from iterating a map; that is, applying a function again and again. Our functions are often polynomials or rational functions and are all defined on the Reimann sphere, which is the plane of complex numbers along with a point at infinity, see Priestley [Pre] for details.

We will start with the simplest definition of a Julia set and work up to a general definition. Along with the Julia set, we will define several related sets.

For a polynomial f the **Julia set** J_f is the boundary of the set of points which go to ∞ under iteration by f. The **Fatou set** F_f is the complement of the Julia set. The **filled in Julia set** K_f is the Julia set unioned with the bounded components of the Fatou set.

Simple examples of a Julia set are not common; as we see in Figure 1, even a simple map like $z^2 - 1$ has a complicated Julia set. So, consider the polynomial z^2. On the real line numbers larger than one go to infinity under iteration. In the complex plane all numbers with length greater than one go to infinity as well since their length is squared after every iteration. This means that the Julia set for z^2 is the circle of radius one centered at the origin. The Fatou set is everything but the circle. The filled in Julia set is the disk of radius one centered at the origin.

In the picture generated by $z^2 - 1$ in Figure 1, the filled-in Julia set is the region colored black, the actual Julia set is the border of the black region(the coloring book outline of the black region), and the Fatou set is the interior of the black region along with all the regions colored other shades of gray.

Unfortunately, the definition that we have used so far only works consistently for polynomials, since Julia sets for more complicated maps often have points that are not on the boundary of the set of points that go to infinity. The general definition requires some complex analysis, but for most maps we discuss the polynomial definition holds. In the general definition we begin by defining the Fatou set, and then defining the Julia set as its complement. See Conway [Co] for a definition of a normal family.

The **Fatou set** F_f of a function f is the set of all points z_0 in the Reimann sphere such that $\{f^n\}_{n=1}^{\infty}$ is a normal family in some neighborhood of z_0. The **Julia set** J_f is the complement of the Fatou set.

Pictures of most Julia sets are complicated and beautiful, as can be seen in figures 1, 2, and 5. Unlike the others, picture five shows only the Julia set.

Dimension

Dimension is a measurement of how complex an object is. We are used to the idea of a point being zero dimensional, a line being one dimensional, and a solid square being two dimensional. However, complicated objects like fractals often fall somewhere in between, as we will see with our example

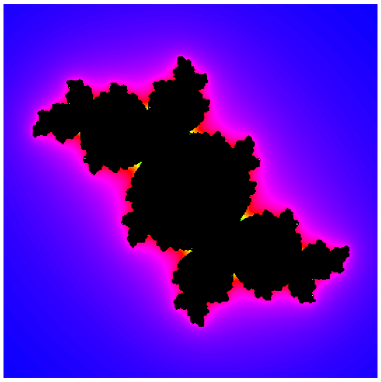

Figure 2: A Julia set often called a "rabbit", $f(z) = z^2 - .12 + .66i$

below.

There are many different notions of dimension. We will define the upper box dimension and will mention the lower box dimension and the Hausdorff dimension. The **upper box dimension** of a set S is

$$\overline{dim}_B S = \overline{\lim_{r \to 0}} \frac{\log N(S,r)}{\log\left(\frac{1}{r}\right)}$$

where $N(S,r)$ is the minimum number of filled squares of side length r required to cover S. The filled squares are commonly called boxes. The lower box dimension is the same as the upper box dimension with the upper limit replaced with a lower limit. Because of this definition, the lower box dimension is always less than or equal to the upper box dimension. When the limit exists in general, the lower and upper box dimensions are equal and the dimension of the object is just called the box dimension.

The Hausdorff dimension is less than or equal to the lower box dimension and is often estimated by calculating or bounding the upper box dimension. Unfortunately the Hausdorff dimension, while a more common dimension to consider mathematically, is much more complicated than either of the box dimensions. For a definition see Pesin, [Pes].

To see how the box dimension works we will look at a simple example that is not a Julia set.

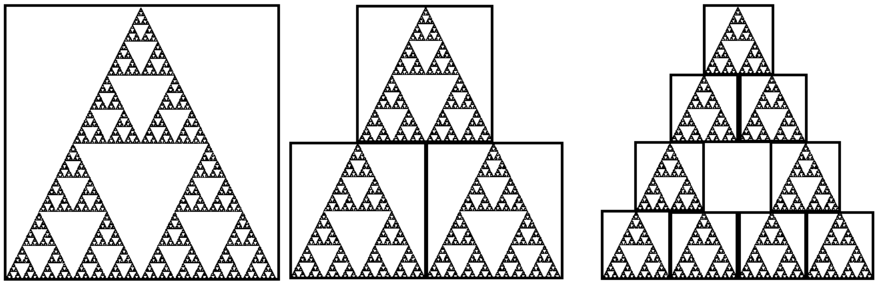

Figure 3: Box coverings for the first three rows of the chart. Note in the rightmost picture that the center box is not needed for the covering.

The Sierpinski gasket

There are many ways to generate pictures of a Sierpinski gasket. A simple way is to start with a solid equilateral triangle. Next, take the midpoints of each side and draw line segments between them, and throw out the inner triangle they create. Now you have three smaller solid triangles. Then, take the midpoints of their sides and repeat the previous process. The Sierpinski gasket is the limit of repeating this process infinitely many times.

We will assume for convenience that the sides of the outer triangle have length one. See Figure 3 for some insight on how the numbers were calculated.

Side length of box	Number of boxes to cover
1	1
1/2	3
1/4	9
1/8	27
$\vdots$	$\vdots$
$1/2^n$	3^n

Box dimension $= \lim_{n\to\infty} \frac{\log 3^n}{\log 2^n} = \frac{\log 3}{\log 2} \approx 1.58.$

We might wonder at this point if the calculation we have completed was quite legal, since we are concerned with the limit overall and not just of one sequence of side lengths. However, this is true because of a theorem sometimes called the Box Counting Theorem that allows us to use a sequence like the one above and states that the dimension calculated by such sequences are equal to the box dimension. Here is the statement of the theorem from Pesin, [Pes]:

Theorem 1 (Box Counting Theorem) *Assume that ϵ_n is a monotonically decreasing sequence of numbers , $\epsilon_n \to 0$, and $\epsilon_{n+1} \geq C\epsilon_n$, for some $C > 0$ which is independent of n. Assume there also exists the limit*

$$\lim_{n\to\infty} \frac{\log N(S, \epsilon_n)}{\log\left(\frac{1}{\epsilon_n}\right)} = d.$$

Then $\underline{dim}_B(S) = \overline{dim}_B(S) = d$.

Porosity

Finding the upper or lower box dimension for something like the Sierpienski gasket is not so difficult. Unfortunately, for most Julia sets the box dimension is not easy to calculate. Most Julia sets have a dimension between zero and two.

One of the tools used to estimate dimension is porosity (also called shallowness). A compact set S is called **porous** if there exists an N such that for any $z \in S$ and $0 < r < 1$, given a square of any side length r containing z and subdivided into N^2 subsquares of side length r/N, then at least one of these squares is disjoint from S . This disjoint square is our "hole" See Figure 4 for an illustration of this property. There is an equivalent definition of porosity using disks rather than squares. This definition basically says that a set is porous if we can find circular holes in our set at every scale.

The definition can be weakened in a couple of different ways as well. The notions of mean porosity and ϵ-mean porosity were developed by Koskela and Rohde [KR] and lose some of the uniformity of the size of the holes within certain limits. The notion of non-uniform porosity was introduced by the

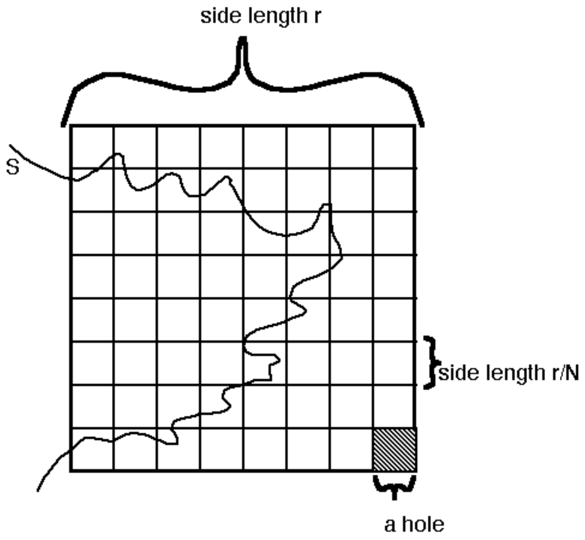

Figure 4: Illustration of definition of porosity.

author [R] and retains the holes at every scale while completely eliminating uniformity.

Porosity and Dimension

Why do we care about porosity? It leads to a bound on the upper box dimension (and so also to the Hausdorff dimension).

Theorem 2 *A porous set in the Reimann sphere has upper box dimension less than 2.*

Proof idea.

This is a sketch of the main ideas of the proof. For a complete proof, see Martio and Vuorinen, [MV].

Let's call our porous set S. We want to estimate $N(S, r)$, the number of boxes needed to cover the set.

First cover S with squares of a side length r, where r is a number greater than zero and less than one. We can assume we have used a finite number of squares since S is compact by the definition of a porous set. Call that number of squares k. Now examine a specific square. Divide the square into N^2 subsquares, where N is the constant coming from porosity. By the definition of porous there must be at least one square that contains no points of S, as in Figure 4 . We do not need it to cover S so throw it out. Repeat this process on the rest of the k squares in our cover. Now we have $k(N^2-1)$ squares of side length $\frac{r}{N}$.

We repeat the process of subdividing our cover of squares of side length $\frac{r}{N}$. After repeating the subdividing and throwing out of squares that do not cover S, we now have $k(N^2-1)^2$ squares of side length $\frac{r}{N^2}$. If we repeat this process n times we have $k(N^2-1)^n$ squares of side length $\frac{r}{N^n}$. For convenience note that $N^2-1 = N^d$ for some $d < 2$.

So the box dimension

$$\overline{dim_B} S = \overline{\lim_{r\to 0}} \frac{\log N(S,r)}{\log\left(\frac{1}{r}\right)} = \overline{\lim_{n\to\infty}} \frac{\log k(N^d)^n}{\log\left(\frac{N^n}{r}\right)}$$

After some calculations we get:

$$\overline{dim_B} S \leq \frac{\log N^{dn}}{\log N^n} = d < 2$$

□

This result implies that the Hausdorff dimension is less than two as well, and that the Lebesgue measure of the Julia set is zero, which follows from the definition of the Hausdorff dimension see Pesin [Pes].

How do we show porosity?

First we must find holes at some point on the Julia set. Once we have porosity for one point in J, we find a conformal mapping that allows us to duplicate these holes at every point.

As Przytycki and Urbański say in [PU], "For rational functions expanding on a Julia set the proof of porosity is easy (it was folklore since a long time). Just pull-back large scale holes to all small scales by iteration of inverse branches of f."

The main difficulty in this approach is keeping the holes reasonably close to the same size while pulling back. If we distort our holes too much under the pull-back we not replicate the porosity.

To keep the distortion of our holes bounded we often use the Koebe One-Quarter Theorem and a related theorem called the Koebe Distortion Theorem.

Theorem 3 (Koebe One-Quarter Theorem) *Let f be a conformal function on the unit disk centered at the origin, with $f(0) = 0$ and $f'(0) = 1$. The image of the unit disk under f contains the open disk of radius $\frac{1}{4}$ centered at the origin.*

A proof this can be found in Carleson and Gamelin, [CG].

We use this to guarantee that we at most have lost three quarters of our original radius. Of course, guaranteeing that we have one constant of porosity for all of the holes is a much more complicated matter, which we will not discuss here.

Julia Sets that are Full of Holes

There are several types of Julia sets that have been shown to be porous. However, all results about porosity of Julia sets have been published in the last ten years. This means the types of the Julia sets that are known to be porous are constrained by complicated conditions to make the mathematics easier. We will consider a few related sets in detail, although we will not discuss the conditions on the sets in detail. The other known results about porosity of Julia sets can be found in [Ge], [Jä], [PR], [PU], [Su], and [Yon].

Our first Julia set that is full of holes is J_P for $P(z) = e^{2\pi i\theta}z + z^2$ where θ is of bounded type. (McMullen, [Mc1]).

Our next set is related to J_P by some complicated means (see [Mc1]). The set J_θ is a subset of the Julia set of $R(z) = e^{2\pi i\tau}z^2\frac{(z-3)}{1-3z}$, where τ is a number related to θ. See Figures 5 and 6. The set J_θ is important because of its relationship both to J_P and to the full Julia set of R. It is porous for

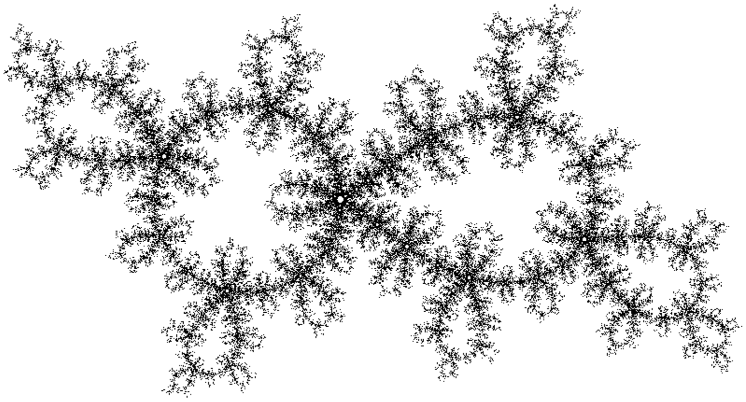

Figure 5: The Julia set for $R(z) = e^{2\pi i\tau} z^2 \frac{(z-3)}{1-3z}$ where τ is a number related to θ and $\theta = \frac{1}{2}(\sqrt{5}-1)$.

irrationals of bounded type and for other irrationals it is a close call: J_θ for $e^{2\pi i\tau(\theta)} z^2 \frac{(z-3)}{1-3z}$, where θ is any irrational, is non-uniformly porous. (Roth, [R])

Despite what is known about J_θ, it is unknown whether J_R is porous. Since mathematically speaking Julia sets are part of a "young" field it is easy to find things that are still yet unknown.

Conclusion

There are many Julia sets that are full of holes. However, there are some that are not. These include the maps for which the Julia set are the whole plane and the more surprising result of Shishikura, [Sh], that there are Julia sets of Hausdorff dimension two.

Also, there are other notions quite similar to but less restrictive than porosity. There is mean porosity and mean ϵ-porosity, both found in Koskela and Rohde, [KR], and non-uniform porosity found in [R]. These can be used in some cases when porosity is not possible to prove.

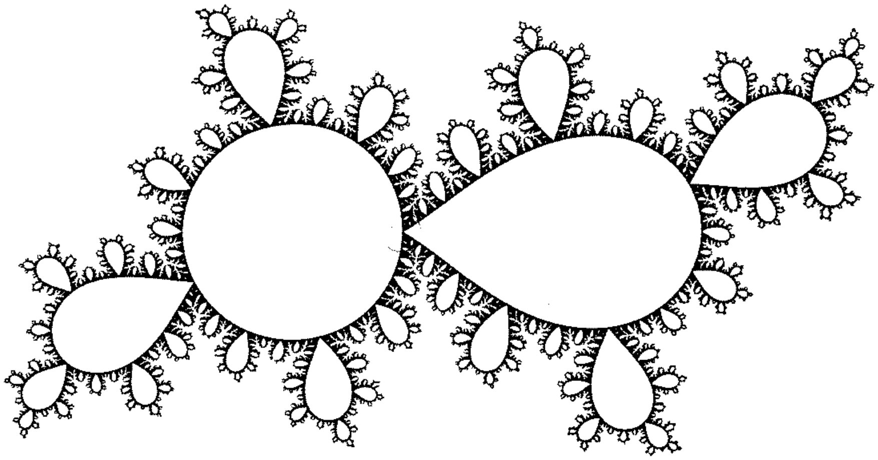

Figure 6: A subset of the previous Julia set, we'll call it J_θ, for $\theta = \frac{1}{2}(\sqrt{5}-1)$ from [Pet].

In the end, we do not know in general which Julia sets are porous and which are not. In fact, for J_R little is known about its dimension or measure. There is much left to explore.

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The figures in the paper were mostly generated using the software of Curt McMullen [Mc2]. Figure 6 comes from Peterson [Pet]. The Sierpinski gasket was generated using BRAZIL Fractal Builder.

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