

## Research Statement for Kimberly A. Roth

My research belongs to the general field of dynamical systems. In particular I study complex dynamics. It is a field whose first explorers included Gaston Julia and Pierre Fatou at the turn of the century, but which made its largest gains in the 1980s with the use of the computers to discover the beauty of Julia sets and the Mandelbrot set.

In complex dynamics, one of the considerations for a Julia set is to quantify its complexity by first showing the Julia set has Lebesgue measure zero. Further description can be given by calculating or finding bounds for the Hausdorff dimension.

I am working with a subset of the Julia set generated by Herman's Blaschke product,  $Q_\theta(z) = e^{2\pi i\tau(\theta)} z^2 \frac{(z-3)}{1-3z}$ , for  $\theta$  irrational. The number  $\tau(\theta)$  is chosen to give the map  $Q_\theta$  the rotation number  $\theta$  when restricted to the unit circle centered at the origin. The subset we work with is  $J_{Q_\theta} \setminus \bigcup_{n=0}^{\infty} (f^{-n}(D))$ , the deleted Julia set, where  $D$  is the open unit disk centered at zero. The deleted Julia set can be related to the Julia set of  $e^{2\pi i\theta} z + z^2$  for irrationals of bounded type.

The goal of my work is to show that the Lebesgue measure of the deleted Julia set is zero, and furthermore that it is non-uniformly porous.

McMullen [7] in 1996 showed porosity for the Julia set of  $e^{2\pi i\theta} z + z^2$  for irrationals of bounded type and I follow a variation of his approach to show the deleted Julia set has measure zero. McMullen followed one of the usual methods to estimate the Hausdorff dimension of a Julia set on the Riemann sphere. By showing that the set is porous, he guarantees that the Hausdorff dimension is less than two. To show porosity for the set, one first shows porosity at a point in the set, then finds appropriate univalent mappings with bounded distortion, and finally uses these maps to get porosity for the whole set.

In my case, there are three main pieces to consider for the deleted Julia set: the unit circle centered at 0, finite preimages of this circle, and limit points of these preimages. On the circle we have porosity at every point with uniform scale. For finite preimages we can use the Koebe distortion theorem to get the same result. So the univalent mapping with bounded distortion needs to be found for each limit point.

To construct the univalent mapping we begin by finding a cone  $K$  around the primary preimage of the circle, in which we can find univalent mappings

with bounded distortion. Once we have this cone, we must show that all but a countable number of points eventually map into the cone under iteration.

Once we have the mapping on all but a countable number of limit points, we can show that almost all points in our deleted Julia set are not Lebesgue density points, which yields the measure zero result. We have also shown that our Julia set is not quite porous, but is non-uniformly porous. This is a stronger condition than measure zero. For example, the rational numbers have measure zero, but are not non-uniformly porous. The fact that the deleted Julia set has measure zero is not new— it was shown for irrationals of bounded type by C. L. Peterson [8] in 1996 and shown by Yampolsky [10] for all irrational  $\theta$  in 1999, using the more complicated techniques of complex bounds.

More details about my research can be found in my dissertation and in my article “Julia Sets that are Full of Holes”, which I have submitted to Mathematics Magazine. A preprint of the article as well as my dissertation can be found on my website.

Next, I plan to write a more expert level article about my research from my dissertation. Further questions I intend to explore include determining what estimates of dimension, if any, come from non-uniform porosity. Also I have indications that the basin of infinity for our Julia set is not a John domain, which is unusual, and I will try to show this as well. Another interesting question to pursue is that of the measure and dimension of the full Julia set for Herman’s Blaschke product. While it has been shown to be locally connected by C. L. Peterson [8] in 1996, no estimates of its measure or dimension are known.

## References

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