

2. $\{ \neg P \wedge \neg Q \} \vdash \neg(P \vee Q)$

1.	$\neg P \wedge \neg Q$	
2.	$P \vee Q$	
3.	P	
4.	$\neg P$	\wedge Elim: 1
5.	$P \wedge \neg P$	\wedge Intro: 3, 4
6.	Q	
7.	$\neg Q$	\wedge Elim: 1
8.	$\neg(P \wedge \neg P)$	
9.	$Q \wedge \neg Q$	\wedge Intro: 6, 8
10.	$\neg\neg(P \wedge \neg P)$	\neg Intro: 8-9
11.	$P \wedge \neg P$	\neg Elim: 10
12.	$P \wedge \neg P$	\vee Elim: 2, 3-5, 6-11
13.	$\neg(P \vee Q)$	\neg Intro: 2-12

Problem 3-47:

1. $\{ \neg(A \vee B) \} \vdash \neg A$

1.	$\neg(A \vee B)$	
2.	A	
3.	$A \vee B$	\vee Intro: 2
4.	$(A \vee B) \wedge \neg(A \vee B)$	\wedge Intro: 3, 1
5.	$\neg A$	\neg Intro: 2-4

2. $\{ \neg(\neg A \wedge B), \neg(\neg B \vee C) \} \vdash A$

1.	$\neg(\neg A \wedge B)$		
2.	$\neg(\neg B \vee C)$		
3.	$\neg A$		
4.	$\neg B$		
5.	$\neg B \vee C$	\vee Intro: 4	
6.	$(\neg B \vee C) \wedge \neg(\neg B \vee C)$	\wedge Intro: 5, 2	
7.	$\neg\neg B$	\neg Intro: 4-6	
8.	B	\neg Elim: 7	
9.	$\neg A \wedge B$	\wedge Intro: 3, 8	
10.	$(\neg A \wedge B) \wedge \neg(\neg A \wedge B)$	\wedge Intro: 9, 1	
11.	$\neg\neg A$	\neg Intro: 11	
12.	A	\neg Elim: 11	

3. $\emptyset \vdash \neg(a = b \wedge b \neq a)$

1.	$a = b \wedge b \neq a$	
2.	$a = b$	\wedge Elim: 1
3.	$b \neq a$	\wedge Elim: 1
4.	$b \neq b$	Ind Id: 3, 2
5.	$\neg(a = b \wedge b \neq a)$	\neg Intro: 1-4

2.4. Chapter 4 Solutions

Problem 4-1:

A	B	$(A \rightarrow B) \wedge (B \rightarrow A)$			$A \leftrightarrow B$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	F
F	F	T	T	T	T

Problem 4-2: Tarski's World Drill

Problem 4-3: 4-3.sen:

1. Tet(c) \rightarrow \neg Tet(a)

2. Tet(b) \rightarrow \neg Tet(d)

3. Tet(b) \rightarrow \neg Tet(c)

4. (Cube(a) \wedge Dodec(b)) \rightarrow LeftOf(a, b)

5. Tet(d) \leftrightarrow Small(d)

6. (Cube(a) \wedge Cube(d)) \rightarrow (LeftOf(a, d) \vee LeftOf(d, a))

7. Cube(d) \leftrightarrow (Medium(d) \vee Large(d))

8. \neg (LeftOf(b, d) \vee RightOf(b, d)) \rightarrow (Tet(b) \vee Tet(d))

9. \neg (Smaller(b, c) \vee Larger(b, c)) \leftrightarrow ((Tet(b) \wedge Dodec(c)) \vee (Tet(c) \wedge Dodec(b)))

Problem 4-4: 4-4.sen:

1. $\text{Tet}(a) \rightarrow \text{FrontOf}(a, d)$

2. $(\text{LeftOf}(a, d) \vee \text{RightOf}(a, d)) \rightarrow \text{Cube}(a)$

3. $\neg \text{Between}(c, a, e) \rightarrow \text{Between}(c, a, d)$
; OR $\text{Between}(c, a, e) \vee \text{Between}(c, a, d)$

4. $\text{Small}(c) \rightarrow \text{RightOf}(c, a)$

5. $\text{RightOf}(c, d) \rightarrow (\text{RightOf}(b, c) \wedge \text{LeftOf}(b, e))$

6. $\text{Tet}(e) \rightarrow (\text{RightOf}(e, b) \leftrightarrow \text{FrontOf}(e, b))$

7. $\text{Cube}(b) \rightarrow (\neg \text{FrontOf}(b, d) \rightarrow \neg \text{BackOf}(b, d))$

8. $\text{BackOf}(c, a) \wedge \text{FrontOf}(c, e)$

9. $\text{FrontOf}(e, d) \vee (\text{Tet}(e) \wedge \text{Large}(e))$
; OR $\neg(\text{Large}(e) \wedge \text{Tet}(e)) \rightarrow \text{FrontOf}(e, d)$

10. $\text{Cube}(a) \vee \text{Cube}(c) \vee \text{Cube}(e)$

11. $\text{Cube}(a) \rightarrow \text{FrontOf}(b, c)$

12. $\text{Larger}(b, a) \wedge \text{Larger}(b, e)$

13. $\text{Larger}(a, c) \wedge \text{Larger}(e, c) \wedge \neg(\text{Large}(a) \vee \text{Large}(e))$

14. $\neg(\text{Larger}(d, b) \vee \text{Smaller}(d, b))$
; OR $(\text{Small}(d) \wedge \text{Small}(b)) \vee (\text{Medium}(d) \wedge \text{Medium}(b)) \vee (\text{Large}(d) \wedge \text{Large}(b))$

15. $\text{Large}(a) \leftrightarrow \text{Cube}(a)$

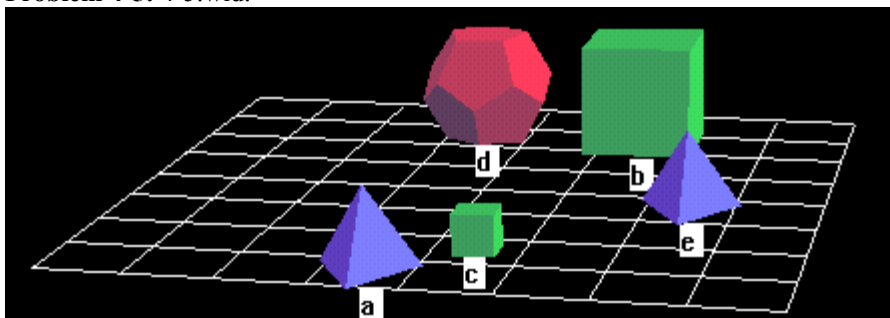
16. $\text{Tet}(e) \vee \text{Cube}(c)$
; OR $\neg \text{Cube}(c) \rightarrow \text{Tet}(e)$

17. $\neg \text{Tet}(e) \rightarrow (\text{Small}(b) \vee \text{Small}(d))$

18. $(\text{Tet}(a) \vee \text{Tet}(c)) \rightarrow (\text{Dodec}(b) \vee \text{Dodec}(d))$

19. $\text{Dodec}(d) \leftrightarrow \text{Cube}(b)$

20. $\text{Dodec}(b) \leftrightarrow \text{Dodec}(c)$

Problem 4-5: 4-5.wld:

Problem 4-6: Tarski's World Drill**Problem 4-8:** 4-8.sen:

1. $\text{Tet}(a) \rightarrow \text{Tet}(b)$

2. $\text{Tet}(b) \rightarrow \text{Tet}(c)$

3. $(\text{Tet}(a) \wedge \text{Tet}(c)) \rightarrow (\text{Large}(a) \vee \text{Large}(c))$

4. $\text{Tet}(a) \wedge \neg \text{Large}(c)$

5. $\text{Small}(c) \rightarrow (\text{Dodec}(d) \rightarrow \neg(\text{Large}(d) \vee \text{Small}(d)))$

6. $\text{Medium}(c) \rightarrow \neg(\text{Cube}(d) \vee \text{Cube}(e) \vee \text{Cube}(f))$

7. $(\text{Dodec}(d) \wedge \text{Small}(d)) \vee \text{Small}(a)$
; OR $\neg \text{Small}(a) \rightarrow (\text{Small}(d) \wedge \text{Dodec}(d))$

8. $\text{Large}(e) \leftrightarrow (\text{Large}(d) \leftrightarrow \text{Large}(f))$

9. $\neg(\text{Larger}(d, e) \vee \text{Smaller}(d, e))$

10. $(\text{Tet}(d) \leftrightarrow \text{Tet}(e)) \vee (\text{Dodec}(d) \leftrightarrow \text{Dodec}(e)) \vee (\text{Cube}(d) \leftrightarrow \text{Cube}(e))$

11. $\text{Large}(f) \rightarrow (\text{Cube}(f) \vee \text{Dodec}(f))$

12. $\text{Larger}(c, e) \rightarrow \text{Larger}(b, c)$

13. $\text{Tet}(a) \wedge \text{Large}(a)$

14. $\text{Tet}(b) \wedge \text{Large}(b)$

15. $\text{Tet}(c) \wedge \text{Medium}(c)$

16. $\text{Dodec}(d) \wedge \text{Small}(d)$

17. $\text{Dodec}(e) \wedge \text{Small}(e)$

18. $\text{Dodec}(f) \wedge \text{Large}(f)$

Problem 4-9:

1. $\text{Gave}(\text{Claire}, \text{Folly}, \text{Max}, 2:03) \rightarrow (\text{Owned}(\text{Claire}, \text{Folly}, 2:00) \wedge \text{Owned}(\text{Max}, \text{Folly}, 2:05))$
2. $\text{Erased}(\text{Max}, \text{Folly}, 2:00) \wedge (\text{Gave}(\text{Max}, \text{Folly}, \text{Claire}, 2:00) \rightarrow \neg \text{Blank}(\text{Folly}, 2:05))$
3. $\neg(\text{Erased}(\text{Max}, \text{Folly}, 2:00) \vee \text{Erased}(\text{Claire}, \text{Folly}, 2:00)) \rightarrow \neg \text{Blank}(\text{Folly}, 2:00)$
4. $\text{Angry}(\text{Max}, 2:05) \rightarrow (\text{Erased}(\text{Claire}, \text{Silly}, 2:00) \vee \text{Erased}(\text{Claire}, \text{Folly}, 2:00))$
5. $\text{Student}(\text{Max}) \leftrightarrow \neg \text{Student}(\text{Claire})$

Problem 4-10:

1. If Max or Claire erased Folly at 2 p.m., then Folly is a disk.
2. Max erased Folly at 2:30 p.m. if and only if Claire erased Silly at 2 p.m.
3. If Folly wasn't blank at 2 p.m., Silly was. OR Silly was blank at 2 p.m. unless Folly was.
4. It is not the case that if Folly wasn't blank at 2 p.m., then Silly was also.

Problem 4-11:

	English	FOL
Names	Abe Stephen Ulysses you me France the treaty Germany Tweedledee a party John Mary the concert	Abe Stephen Ulysses you me France Treaty Germany Tweedledee Party John Mary Concert
Predicates	x can fool y x scratches y x will sign y x gets y x and y went to z together x likes y	CanFool(x, y) Scratch(x, y) Sign(x, y) Gets(x, y) WentTogether(x, y, z) Like(x, y)
Functions	x 's back	back(x)

1. CanFool(Abe, Stephen) \rightarrow CanFool(Abe, Ulysses)
2. Scratch(you, back(me)) \rightarrow Scratch(me, back(you))
3. Sign(France, Treaty) \rightarrow Sign(Germany, Treaty)
4. Gets(Tweedledee, Party) \leftrightarrow Gets(Tweedledum, Party)
5. WentTogether(John, Mary, Concert) \rightarrow (Like(John, Mary) \wedge Like(Mary, John))

A Supplemental Problem: Expressing the basic logic operators using only two (\rightarrow and \neg):

The results:

$\neg P$	$\neg P$
$P \vee Q$	$\neg P \rightarrow Q$
$P \wedge Q$	$\neg(P \rightarrow \neg Q)$
$P \rightarrow Q$	$P \rightarrow Q$

EVEN THOUGH the language of first order logic provides four basic logical symbols (so far), only two of them are truly necessary. This problem shows how to express every type of sentence using only \neg and \rightarrow . The same can be done using (\neg and \wedge) or (\neg and \vee).

Truth tables for the \wedge and \vee :

P	Q	$P \wedge Q$	$\neg(P \rightarrow \neg Q)$	$P \vee Q$	$\neg P \rightarrow Q$
T	T	T	T	T	F T T T
T	F	F	F	T	F T T F
F	T	F	F	T	T F T T
F	F	F	F	F	T F F F

Problem 4-23:

1. Modus Tollens: $\{ A \rightarrow B, \neg B \} \vdash \neg A$

- | | |
|----------------------|--------------------------|
| 1. $A \rightarrow B$ | |
| 2. $\neg B$ | |
| 3. A | |
| 4. B | \rightarrow Elim: 1, 3 |
| 5. $B \wedge \neg B$ | \wedge Intro: 4, 2 |
| 6. $\neg A$ | \neg Intro: 3-5 |

2. Strengthening the Antecedent:

$\{ B \rightarrow C \} \vdash (A \wedge B) \rightarrow C$

- | | |
|---------------------------------|--------------------------|
| 1. $B \rightarrow C$ | |
| 2. $A \wedge B$ | |
| 3. B | \wedge Elim: 2 |
| 4. C | \rightarrow Elim: 1, 3 |
| 5. $(A \wedge B) \rightarrow C$ | \rightarrow Intro: 2-4 |

3. Weakening the Consequent: $\{ A \rightarrow B \} \vdash A \rightarrow (B \vee C)$

- | | |
|-------------------------------|--------------------------|
| 1. $A \rightarrow B$ | |
| 2. A | |
| 3. B | \rightarrow Elim: 1, 2 |
| 4. $B \vee C$ | \wedge Intro: 3 |
| 5. $A \rightarrow (B \vee C)$ | \rightarrow Intro: 2-4 |

4. Constructive Dilemma: $\{ A \vee B, A \rightarrow C, B \rightarrow D \} \vdash C \vee D$

- | | |
|----------------------|--------------------------|
| 1. $A \vee B$ | |
| 2. $A \rightarrow C$ | |
| 3. $B \rightarrow D$ | |
| 4. A | |
| 5. C | \rightarrow Elim: 2, 4 |
| 6. $C \vee D$ | \vee Intro: 5 |
| 7. B | |
| 8. D | \rightarrow Elim: 3, 7 |
| 9. $C \vee D$ | \vee Intro: 8 |
| 10. $C \vee D$ | \vee Elim: 1, 4-6, 7-9 |

5. Trans. of Biconditional: $\{ A \leftrightarrow B, B \leftrightarrow C \} \vdash A \leftrightarrow C$

- | | |
|--------------------------|-----------------------------------|
| 1. $A \leftrightarrow B$ | |
| 2. $B \leftrightarrow C$ | |
| 3. A | |
| 4. B | \leftrightarrow Elim: 1, 3 |
| 5. C | \leftrightarrow Elim 2, 4 |
| 6. C | |
| 7. B | \leftrightarrow Elim: 2, 6 |
| 8. A | \leftrightarrow Elim: 1, 6 |
| 9. $A \leftrightarrow C$ | \leftrightarrow Intro: 3-5, 6-8 |

Problem 4-24:

1. $\emptyset \vdash A \rightarrow (B \rightarrow A)$

- | | |
|--------------------------------------|--------------------------|
| 1. A | |
| 2. B | |
| 3. A | Reit: 1 |
| 4. $B \rightarrow A$ | \rightarrow Intro: 2-3 |
| 5. $A \rightarrow (B \rightarrow A)$ | \rightarrow Intro: 1-4 |

2. $\emptyset \vdash [A \rightarrow (B \rightarrow C)] \leftrightarrow [(A \wedge B) \rightarrow C]$

- | | |
|--|------------------------------------|
| 1. $A \rightarrow (B \rightarrow C)$ | |
| 2. $A \wedge B$ | |
| 3. A | \wedge Elim: 2 |
| 4. B | \wedge Elim: 2 |
| 5. $B \rightarrow C$ | \rightarrow Elim: 1, 3 |
| 6. C | \rightarrow Elim: 5, 4 |
| 7. $(A \wedge B) \rightarrow C$ | \rightarrow Intro: 2-6 |
| 8. $(A \wedge B) \rightarrow C$ | |
| 9. A | |
| 10. B | |
| 11. $A \wedge B$ | \wedge Intro: 9, 10 |
| 12. C | \rightarrow Elim: 8, 11 |
| 13. $B \rightarrow C$ | \rightarrow Intro: 10-12 |
| 14. $A \rightarrow (B \rightarrow C)$ | \rightarrow Intro: 9-13 |
| 15. $[A \rightarrow (B \rightarrow C)] \leftrightarrow [(A \wedge B) \rightarrow C]$ | \leftrightarrow Intro: 1-7, 8-14 |

3. $\{ A \vee (B \wedge C), \neg E, (A \vee B) \rightarrow (D \vee E), \neg A \} \vdash C \wedge D$

Proof Version 1:

- | | |
|--|-----------------------------|
| 1. $A \vee (B \wedge C)$ | |
| 2. $\neg E$ | |
| 3. $(A \vee B) \rightarrow (D \vee E)$ | |
| 4. $\neg A$ | |
| 5. $B \wedge C$ | |
| 6. B | \wedge Elim: 5 |
| 7. $A \vee B$ | \vee Intro: 6 |
| 8. $D \vee E$ | \rightarrow Elim: 3, 7 |
| 9. D | |
| 10. C | \wedge Elim: 5 |
| 11. $C \wedge D$ | \wedge Intro: 9, 10 |
| 12. E | |
| 13. $\neg(C \wedge D)$ | |
| 14. $E \wedge \neg E$ | \wedge Intro: 12, 2 |
| 15. $\neg\neg(C \wedge D)$ | \neg Intro: 13-14 |
| 16. $C \wedge D$ | \neg Elim: 15 |
| 17. $C \wedge D$ | \vee Elim: 8, 9-11, 12-16 |
| 18. A | |
| 19. $\neg(C \wedge D)$ | |
| 20. $A \wedge \neg A$ | \wedge Intro: 18, 4 |
| 21. $\neg\neg(C \wedge D)$ | \neg Intro: 19-20 |
| 22. $C \wedge D$ | \neg Elim: 21 |
| 23. $C \wedge D$ | \vee Elim: 1, 5-17, 18-22 |

3. $\{A \vee (B \wedge C), \neg E, (A \vee B) \rightarrow (D \vee E), \neg A\} \vdash C \wedge D$
Proof Version 2:

1. $A \vee (B \wedge C)$	
2. $\neg E$	
3. $(A \vee B) \rightarrow (D \vee E)$	
4. $\neg A$	
5. A	
6. $\neg(B \wedge C)$	
7. $A \wedge \neg A$	\wedge Intro: 4, 5
8. $\neg\neg(B \wedge C)$	\neg Intro: 6-7
9. $B \wedge C$	\neg Elim: 8
10. $B \wedge C$	
11. $B \wedge C$	Reit: 10
12. $B \wedge C$	\vee Elim: 1, 5-9, 10-11
13. B	\wedge Elim: 12
14. $A \vee B$	\vee Intro: 13
15. $D \vee E$	\rightarrow Elim: 3, 14
16. D	
17. D	Reit: 16
18. E	
19. $\neg D$	
20. $E \wedge \neg E$	\wedge Intro: 18, 2
21. $\neg\neg D$	\neg Intro: 21
22. D	\neg Elim: 22
23. D	\vee Elim: 15, 16-17, 18-22
24. C	\wedge Elim: 12
25. $C \wedge D$	\wedge Intro: 24, 23

Problem 4-25:

1. Logical Equiv: $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$

1. $P \rightarrow Q$	
2. $\neg(\neg P \vee Q)$	
3. $\neg P$	
4. $\neg P \vee Q$	\vee Intro: 3
5. $(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	\wedge Intro: 4, 2
6. $\neg\neg P$	\neg Intro: 3-5
7. P	\neg Elim: 6
8. Q	\rightarrow Elim: 1, 7
9. $\neg P \vee Q$	\vee Intro: 8
10. $(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	\wedge Intro: 9, 2
11. $\neg\neg(\neg P \vee Q)$	\neg Intro: 2-10
12. $\neg P \vee Q$	\neg Elim: 11
13. $\neg P \vee Q$	
14. P	
15. Q	
16. Q	Reit: 15
17. $\neg P$	
18. $\neg Q$	
19. $P \wedge \neg P$	\wedge Intro: 14, 17
20. $\neg\neg Q$	\neg Intro: 18-19
21. Q	\neg Elim: 20
22. Q	\vee Elim: 13, 15-16, 17-21
23. $P \rightarrow Q$	\rightarrow Intro: 14-22
24. $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$	\leftrightarrow Intro: 1-12, 13-23

2. Logical Equiv: $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$

1. $\neg(P \rightarrow Q)$	
2. $\neg(P \wedge \neg Q)$	
3. $\neg(\neg P \vee Q)$	
4. $\neg P$	
5. $\neg P \vee Q$	\vee Intro: 4
6. $(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	\wedge Intro: 5, 3
7. $\neg\neg P$	\neg Intro: 4-6
8. P	\neg Elim: 7
9. Q	
10. $\neg P \vee Q$	\vee Intro: 9
11. $(\neg P \vee Q) \wedge \neg(\neg P \vee Q)$	\wedge Intro: 10, 3
12. $\neg Q$	\neg Intro: 9-11
13. $P \wedge \neg Q$	\wedge Intro: 8, 12
14. $(\neg P \wedge Q) \wedge \neg(\neg P \wedge Q)$	\wedge Intro: 13, 2
15. $\neg\neg(\neg P \vee Q)$	\neg Intro: 3-14
16. $\neg P \vee Q$	\neg Elim: 15
17. P	
18. Q	
19. Q	Reit: 18
20. $\neg P$	
21. $\neg Q$	
22. $P \wedge \neg P$	\wedge Intro: 17, 20
23. $\neg\neg Q$	\neg Intro: 21-22
24. Q	\neg Elim: 23
25. Q	\vee Elim: 16, 18-19, 20-24
26. $P \rightarrow Q$	\rightarrow Intro: 17-25
27. $(P \rightarrow Q) \wedge \neg(P \rightarrow \neg Q)$	\wedge Intro: 26, 1
28. $\neg\neg(P \wedge \neg Q)$	\neg Intro: 2-27
29. $P \wedge \neg Q$	\neg Elim: 28
30. $P \wedge \neg Q$	
31. $P \rightarrow Q$	
32. P	\wedge Elim: 30
33. Q	\rightarrow Elim: 31, 32
34. $\neg Q$	\wedge Elim: 30
35. $Q \wedge \neg Q$	\rightarrow Elim: 33, 34
36. $\neg(P \rightarrow Q)$	\neg Intro: 31-35
37. $\neg(P \rightarrow Q) \leftrightarrow (P \wedge \neg Q)$	\leftrightarrow Intro: 1-29, 30-36