2.5. Chapter 5 Solutions

Problem 5-1: 5-1.sen:

1. ∀x(Cube(x) → Small(x))
2. ∃x Cube(x)
3. ∃v(Cube(v) ∧ Medium(v) ∧ Larger(v, c))
4. ∃u(Small(u) ∧ Cube(u))
5. ∀w (Tet(w) → ∃z (Dodec(z) ∧ Larger(w, z)))
6. ∀x∀y∀z((LeftOf(x, y) ∧ LeftOf(y, z)) → LeftOf(x, z))
7. ∀x ∀y(Larger(x, y) → (Cube(x) ∧ Dodec(y)))
8. ∀x ∀y ((Cube(x) ∧ Cube(y)) → LeftOf(x, y))
9. ∀x (Cube(x) → ∃x Between(x, x, y))

Problem 5-2: 5-2.sen:

1. ∀w(Tet(w) → Large(w))
2. Tet(a) → ∃w Large(w)
3. ∀w Tet(w) → ∃w Large(w)
4. ∀x ∃y ((Cube(x) ∧ Cube(y)) → ¬Larger(x, y))
5. Cube(a)
6. ∀y Tet(y) → ∃x Small(x)
7. ∀x∀y ((Tet(x) ∧ Tet(y)) → ∃z Between(z, x, y))
8. ∀x(Tet(x) ∧ Large(x)) ∧ ∃x∃y LeftOf(x, y)

Problem 5-3: 5-3.sen:

1. Small(a) ∧ Cube(a) ∧ ∃x (FrontOf(a, x)) ∧ Tet(e)
2. Cube(a)
3. ¬∀x BackOf(x, b)
4. Cube(a) ∧ (Cube(b) ∨ Cube(c))
5. Cube(a) ↔ (Cube(b) ↔ Cube(e))
6. ¬ ∀x (Cube (x))
7. ∃x (Cube(x) ∧ Small(x))

Problem 5-3: 5-3.sen continued:

8. ∃x (Cube(x) ∧ Small(x))

9. ∃y (Tet(y) ∧ Large(y))

10. ∀y (Cube(y) → ¬Medium(y))

11. ∀x ((Tet(x) ∧ Small(x)) → FrontOf(x, e))

12. ∀u ((Tet(u) ∧ Medium(u)) → BackOf(u, c))

Problem 5-4: Tarski's World Drill

Problem 5-5: Tarski's World Drill

Problem 5-6: Reserved for Logical Project One (the solution will be provided later)

Problem 5-7: Tarski's World

- *"There is a tetrahedron that is large"* == Sentence 2 (∃x (Tet(x) ∧ Large(x))
- *"There is a cube between a and b"* == Sentence 10 (∃x (Cube(x) ∧ Between(x, a, b)))
- Sentence 5 == "*Some block either is a non-tetrahedron or is large."*
- Sentence 6 == "*Some block either is a non-tetrahedron or is large."*
  ∃x (Tet(x) → Large(x)) ⇔ ∃x(¬Tet(x) ∨ Large(x)) ∴ Sentence 6 expresses the same claim as Sentence 5.
  [Note: for any sentence with the format of ∃x(A(x) → B(x)), one should read it as ∃x(¬A(x) ∨ B(x)).

Problem 5-8: Reserved for Logical Project One (the solution will be provided later)

Problem 5-9: 5-9.sen:

1. ∀x (Tet(x) → Small(x))

2. ∀x (Cube(x) → Small(x))

3. ∀x (Dodec(x) → (Small(x) ∨ Medium(x) ∨ Large(x)))

4. ∃x (Dodec(x) ∧ Large(x))

5. ∃x (Dodec(x) ∧ ¬Large(x))

6. ∃x (Dodec(x) ∧ Small(x))

7. ∃x (Dodec(x) ∧ ¬Small(x))

8. ∃x (Dodec(x) ∧ ¬(Large(x) ∨ Small(x)))

9. ¬∃x (Tet(x) ∧ Large(x))
   ; OR ∀x (Tet(x) → ¬Large(x))

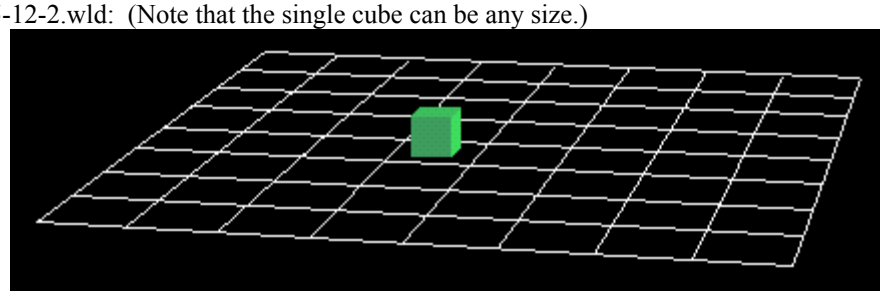
10. ¬∃x (Cube(x) ∧ Large(x))
    ; OR ∀x (Cube(x) → ¬Large(x))

Problem 5-10:

1. ¬∃x (Prime(x) ∧ Even(x))	F
2. ∀x (Prime(x) → (¬Even(x) ∨ x = 2))	T
3. ∃x (Prime(x) ∧ Even(x))	T
4. ∃x (Prime(x) ∧ ¬Even(x))	T

Problem 5-11: Reserved for Logical Project One (the solution will be provided later)

Problem 5-12: Tarski's World
1. Sentence 1 is the correct translation for "Some cube is large." Sentence 4 is the correct translation of "All tetrahedral are small."
2. 5-12-2.wld: (Note that the single cube can be any size.)



3. No – everything in the world has to be a small tetrahedron, so it's not possible to create a world where this case is true but there's exists a tetrahedron that is not small.
4. No – Sentence 1 requires that there be a large dodecahedron. Sentence 2 will always be true in such a world since there is a large object. It will also be true in a world where there's a large object (other than a dodecahedron) or where there is any object other than a dodecahedron; since $\exists x\ (Dodec(x) \rightarrow Large(x)) \Leftrightarrow \exists x\ (\neg Dodec(x) \vee Large(x))$.

Problem 5-14: Reserved for Logical Project One (the solution will be provided later)

Problem 5-15: 5-15.sen:

1. $Tet(b) \wedge Smaller(b, e)$

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2. $\neg\exists x\ (Cube(x) \wedge Medium(x))$
   ; OR $\forall x\ (Cube(x) \rightarrow \neg Medium(x))$

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3. $\neg\exists x\ FrontOf(x, b)$
   ; OR $\forall x\ \neg FrontOf(x, b)$

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4. $\forall x\ (Cube(x) \rightarrow (FrontOf(x, e) \vee BackOf(x, e)))$

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5. $\neg\exists x\ (Cube(x) \wedge Between(x, a, c))$
   ; OR $\forall x\ (Cube(x) \rightarrow Between(x, a, c))$

Problem 5-16:
1. $\forall x\ (Person(x) \rightarrow \neg Disk(x))$
2. $\forall x\ (Disk(x) \rightarrow \neg Person(x))$
3. $\neg\ \exists x\ (Erased(x, Silly, 2{:}00) \vee Erased(x, Silly, 2{:}05))$
4. $\exists t\ (Erased(Claire, Silly, t) \wedge (2{:}00 < t) \wedge (t < 3{:}00))$
5. $\forall x\ (Student(x) \rightarrow \exists y(Disk(y) \wedge Gave(Claire, y, x, 2{:}00))$
6. $\forall x\ ((Disk(x) \wedge Owned(Claire, x, 2{:}00)) \rightarrow Blank(x, 2{:}00))$
7. $Angry(Claire, 3{:}00) \wedge \forall x\ ((Student(x) \wedge Angry(x, 3{:}00)) \rightarrow x = Claire)$
8. $\neg\exists x\ (Person(x) \wedge Erased(x, Folly, 2{:}00))$
9. $\forall x\ ((Person(x) \wedge Erased(x, Silly, 2{:}00)) \rightarrow Angry(x, 2{:}00))$
10. $\forall x\ ((Person(x) \wedge Owned(x, Silly, 2{:}00)) \rightarrow Angry(x, 2{:}05))$

Problem 5-17 :
1. No one owned Folly at 2 p.m.
2. No student erased Folly and was angry at 2 p.m.
3. Anyone to whom Max gave Folly at 2 p.m. was angry at 2:05.
4. Claire never gave Folly to Max.

Problem 5-18:

	English	FOL
Names	I logic company gold you (should be "one") Denmark	I Logic Company Gold One Denmark
Predicates	*x* is brave *x* knows how to forgive *x* is a man *x* is a person *x* is an island *x* cares for *y* *x* is a nation *x* deserves *y* *x* is a certainty *x* is not *y* *x* is *y* *x* is miserable *x* loves *y* *x* glitters *x* is a miller *x* is jolly *x* lived near *y* *x* is a river *x* praises *y* *x* is rotten *x* is in *y*	Brave(x) Forgive(x) Man(x) Person(x) Island(x) Care(x, y) Nation(x) Deserves(x, y) Certainty(x) $x \neq y$ x = y Miserable(x) Loves(x, y) Glitters(x) Miller(x) Jolly(x) LivedNear(x, y) River(x) Praise(x, y) Rotten(x) In(x, y)
Functions	the government of *x* the state of *x*	govt(x) state(x)

1. $\forall x$ (Forgive(x) $\rightarrow$ Brave(x))
2. $\forall x$ (Man(x) $\rightarrow \neg$Island(x))
3. $\neg\exists x$ Care(x, I) $\rightarrow \neg\exists y$ Care(I, y)
4. $\forall x$ (Nation(x) $\rightarrow$ Deserves(x, govt(x))
5. Certainty(Logic) $\wedge \forall x$ (x $\neq$ Logic $\rightarrow \neg$Certainty(x))
6. $\forall x$ ((Person(x) $\wedge$ Miserable(x)) $\rightarrow$ Loves(x, Company))
7. $\neg\forall x$ (Glitters(x) $\rightarrow$ Gold(x)) or $\forall x$(Glitters(x) $\rightarrow \neg$Gold(x))
   (These two sentences are not logically equivalent due to the ambiguity of the sentence.)
8. $\exists x$ (Miller(x) $\wedge$ Jolly(x) $\wedge$ River(Dee) $\wedge$ LivedNear(x, Dee))
9. $\forall x$((Person(x) $\wedge \forall y$ (Person(y) $\rightarrow$ Praise(x, y)) $\rightarrow \forall y$(Person(y) $\rightarrow \neg$Praise(x, y))
10. $\exists x$ (In(x, state(Denmark)) $\wedge$ Rotten(x))

Problem 5-19:

- No Ps are Qs == $\neg\exists x$ (P(x) $\wedge$ Q(x)) or $\forall x$ (P(x) $\rightarrow \neg$Q(x))
- Some Ps are Qs == $\exists x$ (P(x) $\wedge$ Q(x))
- $\neg(\neg\exists x$ (P(x) $\wedge$ Q(x))) $\Leftrightarrow \exists x$ (P(x) $\wedge$ Q(x)) $\neg$ Elim
- $\neg\forall x$(P(x) $\rightarrow \neg$Q(x)) $\Leftrightarrow \exists x \neg$(P(x) $\rightarrow \neg$Q(x)) DeMorgan $\forall$x/$\exists$x Law
  $\Leftrightarrow \exists x \neg(\neg$P(x) $\vee \neg$Q(x)) $\rightarrow$ Elim
  $\Leftrightarrow \exists x$ ($\neg\neg$P(x) $\wedge \neg\neg$Q(x)) DeMorgan
  $\Leftrightarrow \exists x$ (P(x) $\wedge$ Q(x)) $\neg$ Elim

Problem 5-20:

2 ⇔ 4: $\neg\exists y\,(Cube(y) \wedge Large(y)) \Leftrightarrow \forall y\,\neg(Cube(y) \wedge Large(y))$
$\Leftrightarrow \forall y\,(\neg Cube(y) \vee \neg Large(y))$
$\Leftrightarrow \forall x\,(\neg Cube(x) \vee \neg Large(x))$

1 ⇔ 5: $\neg\forall x\,(Cube(x) \rightarrow Small(x)) \Leftrightarrow \exists x\,\neg(Cube(x) \rightarrow Small(x))$
$\Leftrightarrow \exists x\,\neg(\neg Cube(x) \vee Small(x))$
$\Leftrightarrow \exists x\,(\neg\neg\, Cube(x) \wedge \neg\, Small(x))$
$\Leftrightarrow \exists x\,(Cube(x) \wedge \neg Small(x))$
$\Leftrightarrow \exists u\,(Cube(u) \wedge \neg Small(u))$
$\Leftrightarrow \exists u\,(\neg Small(u) \wedge Cube(u))$

3 ⇔ 6: $\neg\forall x\,(Large(x) \leftrightarrow Dodec(x)) \Leftrightarrow \neg\forall x\,((Large(x) \rightarrow Dodec(x)) \wedge (Dodec(x) \rightarrow Large(x)))$
$\Leftrightarrow \exists x\,\neg((Large(x) \rightarrow Dodec(x)) \wedge (Dodec(x) \rightarrow Large(x)))$
$\Leftrightarrow \exists x\,\neg((\neg Large(x) \vee Dodec(x)) \wedge (\neg Dodec(x) \vee Large(x)))$
$\Leftrightarrow \exists x\,(\neg(\neg Large(x) \vee Dodec(x)) \vee \neg(\neg Dodec(x) \vee Large(x)))$
$\Leftrightarrow \exists x\,((\neg\neg Large(x) \wedge \neg Dodec(x)) \vee (\neg\neg Dodec(x) \wedge \neg Large(x)))$
$\Leftrightarrow \exists x\,((Large(x) \wedge \neg Dodec(x)) \vee (Dodec(x) \wedge \neg Large(x)))$

Problem 5-35:

Proof: $\neg\forall x\,P(x) \models \exists x\,\neg P(x)$

1.	$\neg\forall x\,P(x)$	
2.	$\neg\exists x\,\neg P(x)$	
c		
3.	$\neg P(c)$	
4.	$\exists x\,\neg P(x)$	∃ Intro: 3
5.	$\exists x\,\neg P(x) \wedge \neg\exists x\,\neg P(x)$	∧ Intro: 4, 2
6.	$\neg\neg P(c)$	¬ Intro: 3-5
7.	$P(c)$	¬ Elim: 6
8.	$\forall x\,P(x)$	∀ Intro: c-7
9.	$\forall x\,P(x) \wedge \neg\forall x\,P(x)$	∧ Intro: 8, 1
10.	$\neg\neg\exists x\,\neg P(x)$	¬ Intro: 2-9
11.	$\exists x\,\neg P(x)$	¬ Elim: 10

Informally: Suppose $\neg\exists x\,\neg P(x)$ and choose an arbitrary element *c* of the domain. If it can be shown P(c), one can conclude that $\forall x\,P(x)$. To establish P(c), first assume $\neg P(c)$. From this statement one can conclude, by existential generalization, that there does indeed exist an object without property P, or $\exists x\,\neg P(x)$. However, this statement contradicts the first supposition, $\neg\exists x\,\neg P(x)$, so one can conclude that the second supposition $\neg P(c)$ is false. Thus, $\neg\neg P(c)$, and P(c) by double negation elimination. Since *c* is arbitrary, from P(c) one can conclude $\forall x\,P(x)$. However, this conclusion contradicts the premise $\neg\forall x\,P(x)$, so the first supposition must be false. Thus, $\neg\neg\exists x\,\neg P(x)$, or, by double negation, $\exists x\,\neg P(x)$, the desired result. ❑

Problem 5-36:

Corrected:

1.	$\forall x\,[(B(x) \vee T(x)) \rightarrow (M(x) \wedge G(x))]$	
2.	$\forall y\,[(S(y) \vee M(y)) \rightarrow T(y)]$	
3.	$\exists x\,S(x)$	
b		
4.	$S(b)$	
5.	$S(b) \vee M(b)$	∨ Intro: 4
6.	$(S(b) \vee M(b)) \rightarrow T(b)$	∀ Elim: 2
7.	$T(b)$	→ Elim: 6, 5
8.	$(B(b) \vee T(b)) \rightarrow (M(b) \wedge G(b))$	∀ Elim: 1
9.	$B(b) \vee T(b)$	∨ Intro: 7
10.	$M(b) \wedge G(b)$	→ Elim: 8, 9
11.	$M(b)$	∧ Elim: 10
12.	$M(b) \wedge S(b)$	∧ Intro: 11, 4
13.	$\exists x\,(M(x) \wedge S(x))$	∃ Intro: 12
14.	$\exists x\,(M(x) \wedge S(x))$	∃ Elim: 3, b-13

As presented in text: **WRONG!!!**

1.	$\forall x\,[(B(x) \vee T(x)) \rightarrow (M(x) \wedge G(x))]$	
2.	$\forall y\,[(S(y) \vee M(y)) \rightarrow T(y)]$	
3.	$\exists x\,S(x)$	
b		
4.	$S(b)$	
5.	$S(b) \vee M(b)$	∨ Intro: 4
6.	$(S(b) \vee M(b)) \rightarrow T(b)$	∀ Elim: 2
7.	$T(b)$	→ Elim: 6, 5
8.	$(B(b) \vee T(b)) \rightarrow (M(b) \wedge G(b))$	∀ Elim: 1
9.	$B(b) \vee T(b)$	∨ Intro: 7
10.	$M(b)$	*unjustified*
11.	$M(b) \wedge S(b)$	∧ Intro: 11, 4
12.	$\exists x\,(M(x) \wedge S(x))$	∃ Intro: 12
13.	$\exists x\,(M(x) \wedge S(x))$	∃ Elim: 3, b-14

Problem 5-37:

1.	∀x [(B(x) ∧ T(x)) → M(x)]	
2.	∀y [(T(y) ∨ M(y)) → S(y)]	
3.	∃x B(x) ∧ ∃x T(x)	
4.	∃x T(x)	∧ Elim: 3
	[b]	
5.	T(b)	
6.	T(b) ∨ M(b)	∨ Intro: 5
7.	[(T(b) ∨ M(b)) → S(b)]	∀ Elim: 2
8.	S(b)	→ Elim: 7, 6
9.	∃z S(z)	∃ Intro: 8
10.	∃z S(z)	∃ Elim: 4, [b]-9

Problem 5-38:

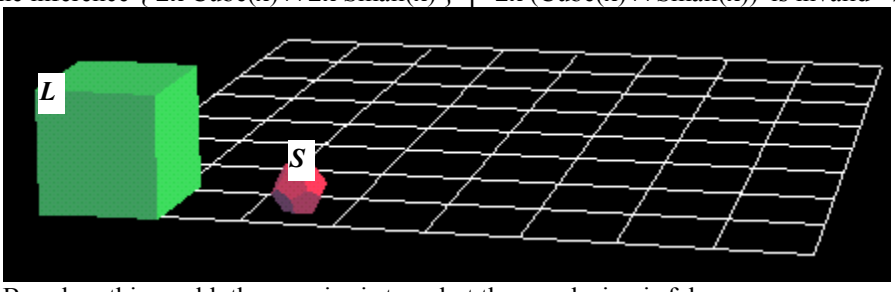
1.	∀y [Cube(y) ∨ Dodec(y)]	
2.	∀x [Cube(x) → Large(x)]	
3.	∃x ¬Large(x)	
	[b]	
4.	¬Large(b)	
5.	Cube(b) ∨ Dodec(b)	∀ Elim: 1
6.	Cube(b)	
7.	Cube(b) → Large(b)	∀ Elim: 2
8.	Large(b)	→ Elim: 7, 6
9.	¬Dodec(b)	
10.	Large(b) ∧ ¬Large(b)	∧ Intro: 8, 4
11.	¬¬Dodec(b)	¬ Intro: 9-10
12.	Dodec(b)	¬ Elim: 11
13.	Dodec(b)	
14.	Dodec(b)	Reit: 13
15.	Dodec(b)	∨ Elim: 5, 6-12, 13-14
16.	∃x Dodec(x)	∃ Intro: 15
17.	∃x Dodec(x)	∃ Elim: 3, [b]-16

Problem 5-39:

1. { ∃x(Cube(x) ∧ Small(x)) } ⊨ ∃x Cube(x) ∧ ∃x Small(x)

1.	∃x(Cube(x) ∧ Small(x))	
	[b]	
2.	Cube(b) ∧ Small(b)	
3.	Cube(b)	∧ Elim: 2
4.	∃x Cube(x)	∃ Intro: 3
5.	Small(b)	∧ Elim: 2
6.	∃x Small(x)	∃ Intro: 5
7.	∃x Cube(x) ∧ ∃x Small(x)	∧ Intro: 4, 6
8.	∃x Cube(x) ∧ ∃x Small(x)	∃ Elim: 1, [b]-7

2. The inference { ∃x Cube(x) ∧ ∃x Small(x) } ⊨ ∃x (Cube(x) ∧ Small(x)) is invalid → see 5-39-2.wld:

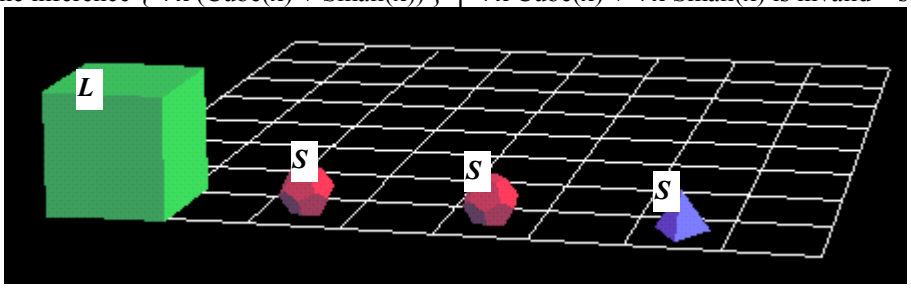


Based on this world, the premise is true, but the conclusion is false.

3. { ∃x Cube(x) ∧ Small(d) } ⊨ ∃x (Cube(x) ∧ Small(d))

1.	∃x Cube(x) ∧ Small(d)	
2.	∃x Cube(x)	∧ Elim: 1
3.	Small(d)	∧ Elim: 1
	[b]	
	4. Cube(b)	
	5. Cube(b) ∧ Small(d)	∧ Intro: 4, 3
	6. ∃x (Cube(x) ∧ Small(d))	∃ Intro: 5
7.	∃x (Cube(x) ∧ Small(d))	∃ Elim: 2, [b]-6

4. The inference { ∀x (Cube(x) ∨ Small(x)) } ⊨ ∀x Cube(x) ∨ ∀x Small(x) is invalid – see 5-39-4.wld:



In this world, the premise (All blocks are either cubes or small.) is true, but the conclusion (Either all blocks are cubes or all blocks are small.) is false.

5. { ∀x Cube(x) ∨ ∀x Small(x) } ⊨ ∀x (Cube(x) ∨ Small(x))

1.	∀x Cube(x) ∨ ∀x Small(x)	
	2. ∀x Cube(x)	
	[b]	
	3. Cube(b)	∀ Elim: 2
	4. Cube(b) ∨ Small(b)	∨ Intro: 3
	5. ∀x (Cube(x) ∨ Small(x))	∀ Intro: [b]-4
	6. ∀x Small(x)	
	[c]	
	7. Small(c)	∀ Elim: 6
	8. Cube(c) ∨ Small(c)	∨ Intro: 7
	9. ∀x (Cube(x) ∨ Small(x))	∀ Intro: [c]-8
10.	∀x (Cube(x) ∨ Small(x))	∨ Elim: 1, 2-5, 6-9

Problem 5-40:

1. { ∀y P(y) } ⊨ ∀x P(x)

1.	∀y P(y)	
	[b]	
	2. P(b)	∀ Elim: 1
3.	∀x P(x)	∀ Intro: [b]-2

2. { ∃y P(y) } ⊨ ∃x P(x)

1.	∃y P(y)	
	[b]	
	2. P(b)	
	3. ∃x P(x)	∃ Intro: 2
4.	∃x P(x)	∃ Elim: 1, [b]-3

Problem 5-41:

1. { ∃x ¬P(x) } ⊨ ¬∀x P(x)

1.	∃x ¬P(x)	
	[b]	
2.	¬P(b)	
3.	∀x P(x)	
4.	P(b)	∀ Elim: 3
5.	P(b) ∧ ¬P(b)	∧ Intro: 4, 2
6.	¬∀x P(x)	¬ Intro: 3-5
7.	¬∀x P(x)	∃ Elim: 1, [b]-6

2. { ∀x ¬P(x) } ⊨ ¬∃x P(x)

1.	∀x ¬P(x)	
2.	∃x P(x)	
	[b]	
3.	P(b)	
4.	¬P(b)	∀ Elim: 1
5.	P(b) ∧ ¬P(b)	∧ Intro: 3, 4
6.	∃x (P(x) ∧ ¬P(x))	∃ Intro: 6
7.	∃x (P(x) ∧ ¬P(x))	∃ Elim: 2, [b]-6
8.	¬∃x P(x)	¬ Intro: 2-7

3. { ¬∃x P(x) } ⊨ ∀x ¬P(x)

1.	¬∃x P(x)	
2.	¬∀x ¬P(x)	
	[b]	
3.	P(b)	
4.	∃x P(x)	∃ Intro: 3
5.	∃x P(x) ∧ ¬∃x P(x)	∧ Intro: 1, 4
6.	¬P(b)	¬ Intro: 3-5
7.	∀x ¬P(x)	∀ Intro: [b]-6
8.	∀x ¬P(x) ∧ ¬∀x ¬P(x)	∧ Intro: 7, 2
9.	¬¬∀x ¬P(x)	¬ Intro: 2-8
10.	∀x ¬P(x)	¬ Elim: 9

2.6. Chapter 6 Solutions

Problems 6-1: Tarski's World Drill

Problem 6-2: 6-2.wld